JOURNAL OF FUNCTIONAL ANALYSIS 110, 434–460 (1992)

Interpretation of the Lavrentiev Phenomenon by Relaxation

GIUSEPPE BUTTAZZO

Istituto di Matematiche Applicate, Via Bonanno, 25/B, 56126 Pisa, Italy

AND

VICTOR J. MIZEL

Department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213

Communicated by H. Brezis

Received July 16, 1991; revised January 6, 1992

We consider functionals of the calculus of variations of the form

$$F(u)=\int_0^1 f(x,u,u')\,dx$$

defined for $u\in W^{1,\infty}(0,1)$, and we show that the relaxed functional $\bar{F}$ with respect to weak $W^{1,1}(0,1)$ convergence can be written as

$$\bar{F}(u)=\int_0^1 f(x,u,u')\,dx+L(u),$$

where the additional term $L(u)$, called the Lavrentiev term, is explicitly identified in terms of F.

1. INTRODUCTION

The term *Lavrentiev phenomenon* refers to a surprising result first demonstrated in 1926 by M. Lavrentiev in [La]. There it was shown that it is possible for the variational integral of a two-point Lagrange problem, which is sequentially weakly lower semicontinuous on the admissible class of absolutely continuous functions, to possess an infimum on the dense subclass of C^1 admissible functions that is *strictly greater* than its minimum value on the full admissible class. Since that time there have been additional works devoted to:

(a) simplifying the original example (Manià [Ma], Heinricher and Mizel [HM1]);

0022-1236/92 $5.00

(b) demonstrating that the phenomenon can occur even with fully regular integrands (Ball and Mizel [BM1, BM2], Davie [Da], Loewen [Lo]);

(c) devising conditions which forestall occurence of the phenomenon (Angell [An], Cesari [Ce], Clarke and Vinter [CV]);

(d) sharpening the specification of the precise dense subclass of admissible functions for which the *Lavrentiev gap* occurs (Ball and Mizel [BM2], Heinricher and Mizel [HM1]);

(e) presenting an analogous gap phenomenon in stochastic control and in certain (deterministic) Bolza problems (Heinricher and Mizel [HM2, HM3]).

Ball and Mizel's investigation [BM2] was undertaken in response to certain previously unresolved foundational questions in nonlinear elasticity. There remains open the question of whether in boundary problems of nonlinear elasticity the presence of Lavrentiev's phenomenon signals the onset of elastic fracture: the force distribution associated with an elastic deformation which provides a global minimum for the elastic energy is then more singular than that associated with minimizers over subclasses of smooth admissible deformations.

The Lavrentiev phenomenon also provides a serious obstacle for numerical schemes of minimization: the cost of any sequence in the smoother admissible class is bounded away from the true minimum value. Furthermore, when a minimizer over the smoother admissible class exists, the approximation scheme typically converges to this suboptimal solution. Ball and Knowles [BK] (see also [Kn, Zo]) have succeeded in the development of numerical approximation schemes which do detect the lower energy *singular minimizers.*

As a simple example of a problem in which the Lavrentiev phenomenon arises, consider the functional

$$F(u)=\int_0^1 (u^2(x)-x)^1\,|u'(x)|^6\,dx$$

over the set

$$\mathscr{A}=(u\in W^{1,1}(0,1)\colon u(0)=0,\, u(1)=s\}.$$

Here (see [Mi1] or [He]) the global minimum over the set $\mathscr{A}$ is given by

$$m_1(s)=\begin{cases}0 & \text{if } |s|\leqslant 1\\ \left(\dfrac{3}{5}\right)^6\left(s^{10}-\dfrac{5}{2}s^8+\dfrac{5}{3}s^6-\dfrac{1}{6}\right) & \text{if } |s|>1,\end{cases}$$

while the infimum over the C^1 or Lipschitz functions in $\mathscr{A}$ is given by

$$m_{Lip}(s)=\left(\frac{3}{5}\right)^6\left(s^{10}-\frac{5}{2}s^8+\frac{5}{3}s^6\right) \qquad \forall s\in\mathbf{R}.$$

The present article revises the above classical view of the phenomenon. Here we adopt the viewpoint that the Lavrentiev gap is actually a relaxation phenomenon assigning to each admissible function u a *Lavrentiev term* $L(u)\geqslant 0$ which specifies the magnitude of the gap between the value of the variational functional itself on u and the smallest sequential lower limit of the values it takes on Lipschitzian admissible functions converging weakly to u. Accordingly, given a sequentially weakly lower semicontinuous (for short "l.s.c.") functional G defined on the class of all admissible functions, we proceed first to examine the functional F which coincides with G on the Lipschitz class but is assigned value $+\infty$ on all non-Lipschitzian admissible functions. We seek the l.s.c. envelope $\bar{F}$ of F (i.e., the maximal sequentially weakly l.s.c. functional dominated by F) on the full class of absolutely continuous admissible functions. Then $L(u)$ is the quantity (nonnegative because of the l.s.c. behavior of G) defined for all admissible functions u by

$$\bar{F}(u)=G(u)+L(u).$$

In Section 2 a characterization of $L(u)$ is provided in terms of the *value function* V associated with the Lagrange problem. This description reveals, in particular, that the Lavrentiev term is local in nature; the quantity $L(u)$ is given as a limiting value of $V(x,u(x))$ as x converges to a critical abscissa for the integrand (Theorem 2.1). This description is then utilized in Section 3 to provide a rather explicit calculation of $L(u)$ for integrands satisfying a homogeneity condition (whose relevance to the Lavrentiev phenomenon was pointed out in Heinricher and Mizel [HM1]) as well as for the far larger class of integrands which only satisfy the homogeneity condition in an asymptotic sense near the relevant critical abscissa. In particular, the integrand presented by Manià [Ma] is fully analyzed by following this approach. Section 4 is devoted to the analysis of the Lavrentiev phenomenon in the case of an integrand which is discontinuous in its arguments; here the Lavrentiev term $L(u)$ is again calculated explicitly. Finally, in Section 5 the Lavrentiev phenomenon is considered in a very general framework. Moreover, the presentation of certain multidimensional problems permits a clear discussion of the Lavrentiev phenomenon for general integral functionals of the calculus of variations.

2. A General Representation of the Lavrentiev Term

In this section we prove a rather general result on the representation of the relaxed functional associated to an integral of the calculus of variations.

Let Ω be the interval $]0, 1[$; we consider the following spaces:

- $W^{1,1}(0, 1)$ the space of all absolutely continuous functions $u: \Omega \to \mathbf{R}$;
- $Lip[0,1]$ the space of all Lipschitz continuous functions $u: \Omega \to \mathbf{R}$;
- $Lip_{loc}]0, 1]$ the space of all functions $u: \Omega \to \mathbf{R}$ which are Lipschitz continuous on every interval $[\delta, 1]$ with $\delta > 0$.

Moreover we set

$$\mathscr{A} = \{u \in W^{1,1}(0, 1) \cap Lip_{loc}]0, 1]: u(0) = 0\}.$$

Let $f: \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a function such that

f is of Carathéodory type (i.e., $f(x, s, z)$ is measurable in x and continuous in (s, z)); (2.1)

$$f(x, s, \cdot) \text{ is convex on } \mathbf{R} \text{ for every } (x, s) \in \Omega \times \mathbf{R}; \tag{2.2}$$

$$f(x, s, 0) = 0 \text{ for every } (x, s) \in \Omega \times \mathbf{R}; \tag{2.3}$$

there exists a function $\omega: \Omega \times \mathbf{R} \times \mathbf{R} \to [0, +\infty[$ with $\omega(x, \tau, t)$ integrable in x and increasing in τ and t such that

$$0 \leqslant f(x, s, z) \leqslant \omega(x, |s|, |z|) \qquad \text{for every} \quad (x, s, z) \in \Omega \times \mathbf{R} \times \mathbf{R}. \tag{2.4}$$

For every $u \in \mathscr{A}$ we define

$$G(u) = \int_0^1 f(x, u, u')\, dx$$

$$F(u) = \begin{cases} G(u) & \text{if } \ u \in Lip[0, 1] \\ +\infty & \text{otherwise} \end{cases}$$

and we denote by $\bar{F}$ the greatest functional on $\mathscr{A}$ which is sequentially l.s.c. with respect to the weak $W^{1,1}(0, 1)$ topology and less than or equal to F. Our goal is to give a representation of $\bar{F}$ on $\mathscr{A}$. Of course, since G is sequentially weakly l.s.c. on $W^{1,1}(0, 1)$ (see for instance Ioffe [Io], or Buttazzo [Bu, Chap. 4,]) we have

$$\bar{F}(u) \geqslant G(u) \qquad \text{for every} \quad u \in \mathscr{A}.$$

Moreover, by the inequality $\bar{F} \leqslant F$ we get

$$\bar{F}(u) = G(u) \qquad \text{for every} \quad u \in Lip[0, 1].$$

In order to characterize the functional $\bar{F}$ on $\mathscr{A}$ we introduce the value function $V(x, s)$ defined for every $(x, s) \in \Omega \times \mathbf{R}$ by

$$V(x, s) = \inf \left\{ \int_0^x f(y, u, u')\, dy \colon u \in Lip[0, x],\, u(0) = 0,\, u(x) = s \right\}$$

and its lower semicontinuous envelope $\bar{V}(x, s)$ with respect to s

$$\bar{V}(x, s) = \liminf_{t \to s} V(x, t).$$

Finally, for every $u \in \mathscr{A}$ we define the "Lavrentiev term"

$$L(u) = \liminf_{x \to 0} \bar{V}(x, u(x)). \tag{2.5}$$

The main result of this section is the following.

THEOREM 2.1. *For every $u \in \mathscr{A}$ we have*

$$\bar{F}(u) = G(u) + L(u).$$

In order to prove Theorem 2.1 we need some preliminary results.

LEMMA 2.2. *Let $u \in \mathscr{A}$ and let $u_h \in Lip[0, 1]$ be such that $u_h(0) = 0$ and $u_h \to u$ weakly in $W^{1,1}(0, 1)$. Then*

$$G(u) + L(u) \leqslant \liminf_{h \to +\infty} F(u_h).$$

Proof. Fix $\delta > 0$; for every $h \in \mathbf{N}$ we have

$$\begin{aligned} F(u_h) &= \int_\delta^1 f(x, u_h, u_h')\, dx + \int_0^\delta f(x, u_h, u_h')\, dx \\ &\geqslant \int_\delta^1 f(x, u_h, u_h')\, dx + V(\delta, u_h(\delta)) \\ &\geqslant \int_\delta^1 f(x, u_h, u_h')\, dx + \bar{V}(\delta, u_h(\delta)). \end{aligned}$$

Passing to the lim inf as $h \to +\infty$ and recalling that the assumptions made on the integrand f provide the weak sequential $W^{1,1}$ lower semicontinuity of the functional $v \mapsto \int_\delta^1 f(x, v, v')\,dx$, we get

$$\liminf_{h \to +\infty} F(u_h) \geqslant \int_\delta^1 f(x, u, u')\,dx + \liminf_{h \to +\infty} \bar{V}(\delta, u_h(\delta))$$
$$\geqslant \int_\delta^1 f(x, u, u')\,dx + \bar{V}(\delta, u(\delta)),$$

where the last inequality follows from the fact that $\bar{V}(x, s)$ is l.s.c. with respect to s. Passing now to the lim inf as $\delta \to 0$, we obtain

$$\liminf_{h \to +\infty} F(u_h) \geqslant \liminf_{\delta \to 0} \int_\delta^1 f(x, u, u')\,dx + \liminf_{\delta \to 0} \bar{V}(\delta, u(\delta))$$
$$= \int_0^1 f(x, u, u')\,dx + L(u) = G(u) + L(u). \quad \blacksquare$$

LEMMA 2.3. *The functional $G + L$ is sequentially l.s.c. on $\mathscr{A}$ with respect to the weak $W^{1,1}(0, 1)$ topology.*

Proof. Take $u, u_h \in \mathscr{A}$ with $u_h \to u$ weakly in $W^{1,1}(0, 1)$; we have to prove that

$$G(u) + L(u) \leqslant \liminf_{h \to +\infty} [G(u_h) + L(u_h)].$$

Without loss of generality, we may assume that the lim inf at the right-hand side is a finite limit. Let $x_h \to 0$ be a sequence such that

$$\bar{V}(x_h, u_h(x_h)) \leqslant L(u_h) + \frac{1}{h} \qquad \text{for every} \quad h \in \mathbf{N}; \tag{2.6}$$

by the definition of $\bar{V}$ and by the properties of f we may find a sequence $s_h \to 0$ such that for every $h \in \mathbf{N}$

$$|s_h - u_h(x_h)| \leqslant \frac{1}{h}, \tag{2.7}$$

$$V(x_h, s_h) \leqslant \bar{V}(x_h, u_h(x_h)) + \frac{1}{h}, \tag{2.8}$$

$$\int_{x_h}^1 f(x, u_h + s_h - u_h(x_h), u_h')\,dx \leqslant \int_{x_h}^1 f(x, u_h, u_h')\,dx + \frac{1}{h}. \tag{2.9}$$

Finally, let $v_h \in Lip[0, x_h]$ be such that

$$v_h(0) = 0, \qquad v_h(x_h) = s_h, \qquad \int_0^{x_h} f(x, v_h, v_h')\, dx \leqslant V(x_h, s_h) + \frac{1}{h}. \tag{2.10}$$

By property (2.3) of f it is easy to see that v_h can be taken monotone; hence, setting

$$w_h = \begin{cases} u_h(x) + s_h - u_h(x_h) & \text{if } x > x_h \\ v_h(x) & \text{if } x \leqslant x_h \end{cases}$$

we have $w_h \in Lip[0, 1]$, $w_h(0) = 0$, and

$$\begin{aligned} \|w_h' - u_h'\|_{L^1(0,1)} &\leqslant \int_0^{x_h} (|v_h'| + |u_h'|)\, dx \\ &= v_h(x_h) + \int_0^{x_h} |u_h'|\, dx = s_h + \int_0^{x_h} |u_h'|\, dx. \end{aligned}$$

Since $s_h \to 0$ and u_h' are equi-integrable on Ω, we get

$$\lim_{h \to +\infty} \|w_h' - u_h'\|_{L^1(0,1)} = 0,$$

so that $w_h \to u$ weakly in $W^{1,1}(0, 1)$. Therefore, by using Lemma 2.2 and (2.6)–(2.10), we obtain

$$\begin{aligned} G(u) + L(u) &\leqslant \liminf_{h \to +\infty} F(w_h) \\ &= \liminf_{h \to +\infty} \left[\int_{x_h}^1 f(x, u_h + s_h - u_h(x_h), u_h')\, dx + \int_0^{x_h} f(x, v_h, v_h')\, dx \right] \\ &\leqslant \liminf_{h \to +\infty} \left[\int_{x_h}^1 f(x, u_h, u_h')\, dx + V(x_h, s_h) + \frac{2}{h} \right] \\ &\leqslant \liminf_{h \to +\infty} \left[G(u_h) + L(u_h) + \frac{4}{h} \right] = \liminf_{h \to +\infty} [G(u_h) + L(u_h)]. \quad \blacksquare \end{aligned}$$

Proof of Theorem 2.1. It is easy to see that

$$L(u) = 0 \qquad \text{for every} \quad u \in Lip[0, 1], u(0) = 0,$$

so that $G + L \leqslant F$ on $\mathscr{A}$. By Lemma 2.3 we have $G + L \leqslant \bar{F}$ on $\mathscr{A}$, and so the proof is achieved if we prove that

$$\bar{F}(u) \leqslant G(u) + L(u) \qquad \text{for every} \quad u \in \mathscr{A}.$$

Let us fix $u \in \mathscr{A}$ and let $x_h \to 0$ be such that

$$L(u) = \lim_{h \to +\infty} \bar{V}(x_h, u(x_h)). \tag{2.11}$$

By the definition of $\bar{V}$ and by the properties of f we may find a sequence $s_h \to 0$ such that for every $h \in \mathbf{N}$

$$|s_h - u(x_h)| \leqslant \frac{1}{h}, \tag{2.12}$$

$$V(x_h, s_h) \leqslant \bar{V}(x_h, u(x_h)) + \frac{1}{h}, \tag{2.13}$$

$$\int_{x_h}^{1} f(x, u + s_h - u(x_h), u')\, dx \leqslant \int_{x_h}^{1} f(x, u, u')\, dx + \frac{1}{h}. \tag{2.14}$$

Finally, let $v_h \in Lip[0, x_h]$ be such that

$$v_h(0) = 0, \qquad v_h(x_h) = s_h, \qquad \int_0^{x_h} f(x, v_h, v_h')\, dx \leqslant V(x_h, s_h) + \frac{1}{h}. \tag{2.15}$$

As in the proof of Lemma 2.3, setting

$$w_h(x) = \begin{cases} u(x) + s_h - u(x_h) & \text{if} \quad x > x_h \\ v_h(x) & \text{if} \quad x \leqslant x_h, \end{cases}$$

we have $w_h \in Lip[0, 1]$, $w_h(0) = 0$, and

$$\lim_{h \to +\infty} \|w_h' - u'\|_{L^1(0,1)} = 0.$$

Hence $w_h \to u$ strongly in $W^{1,1}(0, 1)$ and, by using (2.11)–(2.15), we obtain

$$\begin{aligned}
\bar{F}(u) &\leqslant \liminf_{h \to +\infty} F(w_h) \\
&= \liminf_{h \to +\infty} \left[\int_{x_h}^{1} f(x, u + s_h - u(x_h), u')\, dx + \int_0^{x_h} f(x, v_h, v_h')\, dx \right] \\
&\leqslant \liminf_{h \to +\infty} \left[\int_{x_h}^{1} f(x, u, u')\, dx + V(x_h, s_h) + \frac{2}{h} \right] \\
&\leqslant \liminf_{h \to +\infty} \left[G(u) + \bar{V}(x_h, u(x_h)) + \frac{3}{h} \right] = G(u) + L(u). \quad \blacksquare
\end{aligned}$$

Remark 2.4. It is known that for a two point variational problem the Lavrentiev phenomenon can arise from behavior of the integrand at interior points of the interval of integration as well as at the endpoints [BM2, Da]. However, the perceptive reader will realize that the choice of class $\mathscr{A}$ given before (2.1) has the effect of permitting an analysis via Theorem 2.1 of a possible Lavrentiev gap at any point $(x_0, u_0) \in \Omega \times \mathbf{R}$ in a manner which is unconnected with the presence or absence of such a gap at other points in $\Omega \times \mathbf{R}$. It is simply a matter of translating axes in $\mathbf{R}^2$ (possibly with reflection to include cases with x_0 as the right endpoint of a subinterval) so as to bring (x_0, u_0) to the origin.

One consequence of this approach is that for an integrand f which at all points (x_0, u_0) fails to have a nonzero Lavrentiev term in the sense of Theorem 2.1, there is no function $u \in W^{1,1}(0,1)$ possessing a nonzero $W^{1,1}$–$W^{1,\infty}$ Lavrentiev term. In the general case, the analysis of this Lavrentiev term can actually be carried out by use of Theorem 2.1 whenever the function u has finitely many singular points. On the other hand, we do not know the form of the $W^{1,1}$–$W^{1,\infty}$ Lavrentiev term $L(u)$ in the case of arbitrary $u \in W^{1,1}(0, 1)$.

3. Some Particular Cases

In this section we discuss some particular cases in which the expression of the Lavrentiev term $L(u)$ can be reduced to a simpler form. To begin with, let us consider an integrand f satisfying conditions (2.1)–(2.4) and the following invariance property (see Heinricher and Mizel [HM1]):

there exists $\gamma \in]0, 1[$ such that for every $t > 0$ and $(x, s, z) \in \Omega \times \mathbf{R} \times \mathbf{R}$

$$tf(tx, t^\gamma s, t^{\gamma-1} z) = f(x, s, z). \tag{3.1}$$

In this case the following proposition holds.

Proposition 3.1. *For every $u \in \mathscr{A}$*

$$L(u) = \liminf_{x \to 0} \bar{V}\left(1, \frac{u(x)}{x^\gamma}\right).$$

Proof. Let us fix $(x, s) \in \Omega \times \mathbf{R}$ and $u \in Lip[0, x]$ with $u(0) = 0$ and $u(x) = s$. Setting $y = tx$ and $v(t) = x^{-\gamma} u(tx)$ we get

$$\begin{aligned}\int_0^x f(y, u(y), u'(y))\, dy &= \int_0^1 xf(xt, u(xt), u'(xt))\, dt \\ &= \int_0^1 xf(xt, x^\gamma v(t), x^{\gamma-1} v'(t))\, dt \\ &= \int_0^1 f(t, v(t), v'(t))\, dt.\end{aligned}$$

Therefore

$$V(x, s) = \inf\{F(v) : v \in Lip[0, 1], v(0) = 0, v(1) = sx^{-\gamma}\} = V(1, sx^{-\gamma})$$

and the conclusion follows from formula (2.5) for the Lavrentiev term. ∎

EXAMPLE 3.2. Let $p > 1$, $\alpha \in]0, 1[$, and let

$$f(x, s, z) = |s - x^{\alpha}|\ |z|^{p}.$$

It is easy to see that, if $\alpha = (p-1)/(p+1)$, then f satisfies all conditions (2.1)–(2.4) and the invariance condition (3.1) with $\gamma = \alpha$. By Heinricher and Mizel [HM1], for every $s \in \mathbf{R}$ we have

$$\inf\{F(v) : v \in Lip[0, 1], v(0) = 0, v(1) = s\} = G(u_s),$$

where u_s is the function

$$u_s(x) = sx^{\beta} \qquad \left(\beta = \frac{p}{p+1}\right).$$

Therefore, an easy calculation gives

$$V(1, s) = \begin{cases} \beta^{p-1} s^{p}(1 - \beta s) & \text{if } s \leqslant 1 \\ \beta^{p-1}[2(1-\beta) - s^{p}(1 - \beta s)] & \text{if } s > 1. \end{cases}$$

Note that in this case, if $u(x) = x^{\alpha}$, we have $G(u) = 0$ whereas

$$L(u) = V(1, 1) = \beta^{p-1}(1 - \beta).$$

We consider now a larger class of integrands which only satisfy the homogeneity condition in an asymptotic sense near the relevant singular abscissa. Let $p > 1$, let $\alpha \in [1, p[$, and suppose the integrand $f : \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ has the form

$$f(x, s, z) = x^{\alpha - 1} a(x, s)\ |z|^{p},$$

where $a(x, s)$ is a nonnegative continuous function such that, setting $\gamma = (p - \alpha)/p$, for every $y \in \Omega$ the functions $m_y, M_y : \mathbf{R} \to [0, +\infty]$ defined by

$$m_y(s) = \inf\{a(x, x^{\gamma}s) : x \leqslant y\}$$

$$M_y(s) = \sup\{a(x, x^{\gamma}s) : x \leqslant y\}$$

are locally bounded.

For every $x, y \in \Omega$ with $x \leqslant y$ we consider the functionals

$$F_x(u) = \int_0^x f(t, u, u')\, dt$$

$$F_{*x,y}(u) = \int_0^x t^{\alpha-1} m_y(t^{-\gamma} u)\, |u'|^p\, dt$$

$$F^*_{x,y}(u) = \int_0^x t^{\alpha-1} M_y(t^{-\gamma} u)\, |u'|^p\, dt$$

and the respective value functions

$$V(x, s) = \inf\{F_x(u): u \in \mathscr{A}(x, s)\}$$
$$V_*(x, y, s) = \inf\{F_{*x,y}(u): u \in \mathscr{A}(x, s)\}$$
$$V^*(x, y, s) = \inf\{F^*_{x,y}(u): u \in \mathscr{A}(x, s)\},$$

where $\mathscr{A}(x, s)$ is the set

$$\mathscr{A}(x, s) = \{u \in Lip[0, x]: u(0) = 0, u(x) = s\}.$$

It is immediately seen that for every $s \in \mathbf{R}$ and every $x, y \in \Omega$ with $x \leqslant y$

$$V_*(x, y, s) \leqslant V(x, s) \leqslant V^*(x, y, s). \tag{3.2}$$

Hereafter we shall suppress the parameter y in expressions such as $V_*(x, y, s)$ and $V^*(x, y, s)$ when no confusion can arise.

We now proceed to evaluate the functions V_* and V^* by using a verification argument based on the study of variational problems of the form

$$\inf\left\{\int_0^x t^{\alpha-1} m(t^{-\gamma} u)\, |u'|^p\, dt\right\}, \tag{3.3}$$

where $m: \mathbf{R} \to \mathbf{R}$ is a locally bounded Borel function. If $I(u)$ denotes the integral in (3.3) and $W(x, s)$ is its value function, we will show that

$$W(x, s) = \inf\{I(u): u \in \mathscr{A}(x, s)\} = I(u_0),$$

where $u_0(t) = (t/x)^{p\gamma/(p-1)} s$. Indeed, setting for simplicity $k = p\gamma/(p-1)$, the following proposition holds.

PROPOSITION 3.3. *The function*

$$h(S) = pk^{p-1} \left| \int_0^S m(\xi)\, |\xi|^{p-1}\, d\xi \right|$$

is the solution of the Hamilton–Jacobi equation

$$\begin{cases}\gamma S h'(S) = \sup\{Qh'(S) - m(S)\,|Q|^p \colon Q \in \mathbf{R}\} \\ h(0) = 0\end{cases} \tag{3.4}$$

and for every $(x, s) \in \Omega \times \mathbf{R}$

$$W(x, s) = h(x^{-\gamma} s) = I(u_0).$$

Proof. By explicitly carrying out the maximization, the Hamilton–Jacobi equation (3.4) becomes

$$\begin{cases}h'(S) = pk^{p-1} m(S)\,|S|^{p-2} S \\ h(0) = 0,\end{cases}$$

that is,

$$h(S) = pk^{p-1} \left| \int_0^S m(\xi)\,|\xi|^{p-1}\,d\xi \right|.$$

Now let $u \in \mathscr{A}(x, s)$; from (3.4), taking $S(t) = t^{-\gamma} u(t)$ and $Q(t) = t^{1-\gamma} u'(t)$, we have

$$\begin{aligned} t^{-1} m(S(t))\,|Q(t)|^p &\geqslant t^{-1} h'(S(t))(Q(t) - \gamma S(t)) \\ &= h'(S(t))\,S'(t) = (h \circ S)'\,(t), \end{aligned}$$

where the last equality follows from the chain rule for composition with Lipschitz functions (see for instance Marcus and Mizel [MM1]). Integrating on $]0, x[$ yields

$$I(u) \geqslant \int_0^x (h \circ S)'\,(t)\,dt = h(S(x)) - \lim_{t \to 0} h(S(t)) = h(x^{-\gamma} s), \tag{3.5}$$

where we have used the fact that $u \in Lip[0, x]$ implies that

$$\lim_{t \to 0} t^{-\gamma} u(t) = 0.$$

Taking the infimum on u in (3.5) we obtain

$$W(x, s) \geqslant h(x^{-\gamma} s).$$

On the other hand, the functions

$$u_\varepsilon(t) = \begin{cases} (t/x)^k s & \text{if} \quad t \geqslant \varepsilon \\ t s \varepsilon^{k-1}/x^k & \text{if} \quad t < \varepsilon \end{cases}$$

belong to $\mathscr{A}(x, s)$, so that for ε small enough

$$W(x, s) \leqslant I(u_\varepsilon) = \int_0^\varepsilon t^{\alpha-1} m(t^{-\gamma} u_\varepsilon)\, |u'_\varepsilon|^p\, dt + \int_\varepsilon^x t^{\alpha-1} m(t^{-\gamma} u_0)\, |u'_0|^p\, dt.$$

Passing to the limit as $\varepsilon \to 0$ it is easily seen that the first integral goes to 0, hence

$$W(x, s) \leqslant I(u_0).$$

An easy calculation shows that $I(u_0) = h(x^{-\gamma} s)$, and this achieves the proof. ∎

We can now evaluate the functions V_* and V^*. From Proposition 3.3 we get

$$V_*(x, s) = h_*(x^{-\gamma} s) \qquad V^*(x, s) = h^*(x^{-\gamma} s),$$

where

$$h_*(S) = pk^{p-1} \left| \int_0^S m_y(\xi)\, |\xi|^{p-1}\, d\xi \right|$$

$$h^*(S) = pk^{p-1} \left| \int_0^S M_y(\xi)\, |\xi|^{p-1}\, d\xi \right|.$$

Therefore, from inequalities (3.2), since the functions h_* and h^* are continuous,

$$h_*(x^{-\gamma} s) \leqslant \bar{V}(x, s) \leqslant h^*(x^{-\gamma} s). \tag{3.6}$$

Recalling Theorem 2.1, formula (3.6) yields for every $u \in \mathscr{A}$

$$\liminf_{x \to 0^+} h_*(x^{-\gamma} u(x)) \leqslant L(u) \leqslant \liminf_{x \to 0^+} h^*(x^{-\gamma} u(x)). \tag{3.7}$$

Finally, taking in (3.7) the limit as $y \to 0$, and applying the monotone convergence theorem, we obtain the following result.

THEOREM 3.4. *Under the assumptions above on $f(x, s, z)$, for every $u \in \mathscr{A}$ we have*

$$pk^{p-1} \liminf_{x \to 0^+} \left| \int_0^{x^{-\gamma} u(x)} m_0(\xi)\, |\xi|^{p-1}\, d\xi \right|$$

$$\leqslant L(u) \leqslant pk^{p-1} \liminf_{x \to 0^+} \left| \int_0^{x^{-\gamma} u(x)} M_0(\xi)\, |\xi|^{p-1}\, d\xi \right|,$$

where the functions m_0, M_0 *are given by*

$$m_0(s) = \sup\{m_y(s): y > 0\} = \lim_{y \to 0^+} m_y(s)$$

$$M_0(s) = \inf\{M_y(s): y > 0\} = \lim_{y \to 0^+} M_y(s).$$

Remark 3.5. The same sort of analysis can be carried out whenever

$$f(x, s, z) = x^{-1} a(x, s)\, \varphi(x^{1-\gamma} z),$$

where φ is a nonnegative superlinear convex function satisfying $\varphi(0) = 0$. In the case considered here $\varphi(z) = |z|^p$.

EXAMPLE 3.6. Consider the functional studied by Manià (see for instance Manià [Ma], Cesari [Ce])

$$F(u) = \int_0^1 (u^3 - x)^2 \, |u'|^p \, dx \qquad (p > 3).$$

The integrand f has the form

$$f(x, s, z) = (s^3 - x)^2 \, |z|^p = x^2 (s^3 x^{-1} - 1)^2 \, |z|^p$$

so that $\alpha = 3$, $\gamma = (p-3)/p$, and $a(x, s) = (s^3 x^{-1} - 1)^2$.

When $p > 9/2$, which corresponds to $\gamma > 1/3$, one finds easily

$$m_0(s) = M_0(s) = 1 \qquad \text{for every} \quad s \in \mathbf{R}.$$

Therefore, from Theorem 3.4

$$L(u) = \left(\frac{p-3}{p-1}\right)^{p-1} \liminf_{x \to 0^+} \frac{|u(x)|^p}{x^{p-3}}.$$

In particular, $L(u) = +\infty$ if $u(x) = x^{1/3}$.

When $p = 9/2$, which corresponds to $\gamma = 1/3$, one has

$$m_0(s) = M_0(s) = (s^3 - 1)^2 \qquad \text{for every} \quad s \in \mathbf{R}$$

whence

$$L(u) = \frac{1}{35} \left(\frac{3}{7}\right)^{7/2} \liminf_{x \to 0^+} H(Z(x))$$

with $Z(x) = u^3(x)/x$ and $H(Z) = |Z|^{3/2} (15Z^2 - 42Z + 35)$. In particular, if $u(x) = x^{1/3}$

$$L(u) = \frac{8}{35} \left(\frac{3}{7}\right)^{7/2}.$$

When $p \in]3, 9/2[$, which corresponds to $\gamma < 1/3$, Theorem 3.4 does not apply because the functions m_y and M_y are locally bounded. However, it is possible to show that in this case the Lavrentiev phenomenon does not occur, that is,

$$L(u) = 0 \qquad \text{whenever} \quad \int_0^1 f(x, u, u')\, dx < +\infty.$$

Indeed, if $u \in \mathscr{A}$ and

$$\lim_{\varepsilon \to 0} \frac{|u(x_\varepsilon)|^{p+6}}{x_\varepsilon^{p-1}} = 0$$

for a suitable sequence $x_\varepsilon \to 0$, taking

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if} \quad x \geqslant x_\varepsilon \\ xu(x_\varepsilon)/x_\varepsilon & \text{if} \quad x < x_\varepsilon \end{cases}$$

we get

$$\begin{aligned} L(u) &\leqslant \liminf_{\varepsilon \to 0} \int_0^{x_\varepsilon} \left(x^3 \frac{u^3(x_\varepsilon)}{x_\varepsilon^3} - x \right)^2 \left| \frac{u(x_\varepsilon)}{x_\varepsilon} \right|^p dx \\ &= \lim_{\varepsilon \to 0} \left[\frac{|u(x_\varepsilon)|^{p+6}}{7x_\varepsilon^{p-1}} + \frac{|u(x_\varepsilon)|^p}{3x_\varepsilon^{p-3}} - \frac{2\,|u(x_\varepsilon)|^{p+3}}{5x_\varepsilon^{p-2}} \right] = 0. \end{aligned}$$

On the contrary, if there exists $c > 0$ such that

$$|u(x)| \geqslant c x^{(p-1)/(p+6)} \tag{3.8}$$

for all x small enough, we have

$$\begin{aligned} \int_0^\varepsilon f(x, u, u')\, dx &\geqslant \frac{1}{2} \int_0^\varepsilon c^6 x^{6(p-1)/(p+6)}\, |u'|^p\, dx \\ &\geqslant \frac{c^6}{2}\, \varepsilon y_\varepsilon^{6(p-1)/(p+6)}\, |u'(y_\varepsilon)|^p \\ &\geqslant \frac{c^6}{4}\, \varepsilon y_\varepsilon^{6(p-1)/(p+6)} \left| \frac{u(x_\varepsilon)}{x_\varepsilon} \right|^p \end{aligned}$$

for suitable $0 < x_\varepsilon < y_\varepsilon < \varepsilon$. Therefore,

$$\int_0^\varepsilon f(x, u, u')\, dx \geqslant \frac{c^6}{4} \left| \frac{u(x_\varepsilon)}{x_\varepsilon^{(p-1)/(p+6)}} \right|^p$$

which is in contradiction with (3.8) if $f(x, u, u') \in L^1(\Omega)$.

4. An Example with a Discontinuous Integrand

Let us fix a real number $p>1$ and a function $\varphi \in W^{1,1}(0,1)$ such that $\varphi(0)=0$ and $\varphi \in W^{1,p}(\delta, 1)$ for every $\delta>0$. Define the mappings a_φ: $]0,1[\times \mathbf{R} \to \mathbf{R}$ and F: $W^{1,1}(0,1) \to [0,+\infty]$ by

$$a_\varphi(x,s)=\begin{cases} 0 & \text{if } \ s=\varphi(x) \\ 1 & \text{if } \ s\neq \varphi(x) \end{cases}$$

$$F(u)=\begin{cases} \int_0^1 a_\varphi(x,u)\,|u'|^p\,dx & \text{if } \ u\in W^{1,\infty}(0,1),\, u(0)=0 \\ +\infty & \text{otherwise,} \end{cases}$$

and consider the relaxed functional $\bar{F}$: $W^{1,1}(0,1) \to [0,+\infty]$ defined by

$$\bar{F}=\sup\{G\colon W^{1,1}(0,1)\to[0,+\infty]\colon G\leqslant F,\ G \text{ sequentially weakly l.s.c.}\}.$$

The main result of this section is the following.

THEOREM 4.1. *For every $u \in W^{1,1}(0,1)$ with $u(0)=0$ we have*

$$\bar{F}(u)=\int_0^1 a_\varphi(x,u)\,|u'|^p\,dx+\liminf_{x\to 0^+}\frac{|u(x)|^p\wedge|\varphi(x)|^p}{x^{p-1}}. \tag{4.1}$$

The proof of Theorem 4.1 will be obtained by means of some preliminary lemmas. Let us define $\mathscr{A}(0,1)=\{u\in W^{1,1}(0,1)\colon u(0)=0\}$ and, for every $u\in\mathscr{A}(0,1)$

$$G(u)=\int_0^1 a_\varphi(x,u)\,|u'|^p\,dx$$

$$L(u)=\liminf_{x\to 0^+}\frac{|u(x)|^p\wedge|\varphi(x)|^p}{x^{p-1}}$$

$$F_p(u)=\begin{cases} G(u) & \text{if } \ u\in W^{1,p}(0,1) \\ +\infty & \text{otherwise} \end{cases}$$

Since

$$\lim_{x\to 0^+}\frac{|u(x)|^p}{x^{p-1}}=0 \qquad \text{for every} \quad u\in W^{1,p}(0,1) \text{ with } u(0)=0, \tag{4.2}$$

we have

$$G\leqslant G+L\leqslant F_p\leqslant F \qquad \text{on} \quad \mathscr{A}(0,1).$$

Moreover, since G is sequentially weakly $W^{1,1}(0,1)$-l.s.c.,

$$G\leqslant \bar{F}_p\leqslant \bar{F} \qquad \text{on} \quad \mathscr{A}(0,1). \tag{4.3}$$

LEMMA 4.2. *Let* $u \in \mathscr{A}(0,1)$ *be such that* $G(u) < +\infty$. *Then* $u \in W^{1,p}(\delta, 1)$ *for every* $\delta > 0$.

Proof. Setting $E = \{x \in]0, 1[: u(x) = \varphi(x)\}$, for every $\delta > 0$ we get

$$\begin{aligned}\int_\delta^1 |u'|^p\,dx &= \int_{]\delta,1[\,\cap E} |u'|^p\,dx + \int_{]\delta,1[\,\setminus E} |u'|^p\,dx \\ &= \int_{]\delta,1[\,\cap E} |\varphi'|^p\,dx + \int_{]\delta,1[\,\setminus E} a_\varphi(x,u)\,|u'|^p\,dx \\ &\leqslant \int_\delta^1 |\varphi'|^p\,dx + G(u) < +\infty.\end{aligned}$$

Therefore $u \in W^{1,p}(\delta, 1)$. ∎

LEMMA 4.3. *For every* $u \in \mathscr{A}(0,1)$ *and every* $\varepsilon > 0$ *there exists* $u_\varepsilon \in W^{1,\infty}(0,1)$ *such that* $u_\varepsilon(0) = 0$, $u_\varepsilon \to u$ *strongly in* $W^{1,1}(0,1)$, *and*

$$\liminf_{\varepsilon \to 0^+} F(u_\varepsilon) \leqslant G(u) + \liminf_{x \to 0^+} \frac{|u(x)|^p}{x^{p-1}}. \tag{4.4}$$

Proof. Let $u \in \mathscr{A}(0,1)$ be such that the right-hand side of (4.4) is finite; then, by Lemma 4.2, $u \in W^{1,p}(\delta, 1)$ for every $\delta > 0$. Let $x_\varepsilon \to 0$ be such that

$$\lim_{\varepsilon \to 0^+} \frac{|u(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} = \liminf_{x \to 0^+} \frac{|u(x)|^p}{x^{p-1}}. \tag{4.5}$$

It is known (see for instance Liu [Li] or Marcus and Mizel [MM2, Lemma 1]) that for every $\varepsilon > 0$ there exist an open subset A_ε of $]x_\varepsilon, 1[$ and a Lipschitz function v_ε (actually v_ε can be taken in $C^1(\mathbf{R})$) such that

$$\operatorname{meas}(A_\varepsilon) \leqslant \varepsilon \qquad v_\varepsilon = u \text{ on }]x_\varepsilon, 1[\,\setminus A_\varepsilon, \qquad \|v_\varepsilon - u\|_{W^{1,p}(x_\varepsilon, 1)} \leqslant \varepsilon.$$

Moreover, possibly refining the sequences (A_ε) and (v_ε) we may also assume that

$$\int_{A_\varepsilon} |v_\varepsilon'|^p\,dx \leqslant \varepsilon \qquad \text{and} \qquad \left| \frac{|v_\varepsilon(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} - \frac{|u(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} \right| \leqslant \varepsilon. \tag{4.6}$$

Define now

$$u_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{if } \; x > x_\varepsilon \\ \dfrac{v_\varepsilon(x_\varepsilon)}{x_\varepsilon}\,x & \text{if } \; x \leqslant x_\varepsilon. \end{cases}$$

We have $u_\varepsilon \in W^{1,\infty}(0,1)$, $u_\varepsilon \to u$ strongly in $W^{1,1}(0,1)$, and

$$F(u_\varepsilon) = \int_0^{x_\varepsilon} a_\varphi(x, u_\varepsilon)\,|u'_\varepsilon|^p\,dx + \int_{A_\varepsilon} a_\varphi(x, u_\varepsilon)\,|u'_\varepsilon|^p\,dx$$

$$+ \int_{]x_\varepsilon, 1[\setminus A_\varepsilon} a_\varphi(x, u_\varepsilon)\,|u'_\varepsilon|^p\,dx$$

$$\leqslant \frac{|v_\varepsilon(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} + \int_{A_\varepsilon} |v'_\varepsilon|^p\,dx + \int_{x_\varepsilon}^1 a_\varphi(x, u)\,|u'|^p\,dx.$$

Passing to the limit as $\varepsilon \to 0^+$, and recalling (4.5) and (4.6), we obtain (4.4). ∎

Remark. 4.4. From Lemma 4.3 we obtain immediately

$$\bar{F}(u) \leqslant G(u) + \liminf_{x\to 0^+} \frac{|u(x)|^p}{x^{p-1}} \qquad \text{for every} \quad u \in \mathscr{A}(0,1). \tag{4.7}$$

Therefore, by (4.2) and (4.7) we have

$$\bar{F} \leqslant F_p \qquad \text{on} \quad \mathscr{A}(0,1).$$

Hence $\bar{F} \leqslant \bar{F}_p$ which, together with (4.3) gives

$$\bar{F} = \bar{F}_p \qquad \text{on} \quad \mathscr{A}(0,1).$$

Thus, in what follows, we shall use the functional F_p instead of F; this allows us to use $W^{1,p}$ functions instead of Lipschitz functions in the approximations.

LEMMA 4.5. *For every $u \in \mathscr{A}(0,1)$ and every $\varepsilon > 0$ there exists $u_\varepsilon \in W^{1,p}(0,1)$ such that $u_\varepsilon(0) = 0$, $u_\varepsilon \to u$ strongly in $W^{1,1}(0,1)$, and*

$$\liminf_{\varepsilon\to 0^+} F_p(u_\varepsilon) \leqslant G(u) + \liminf_{x\to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}}. \tag{4.8}$$

Proof. Let $u \in \mathscr{A}(0,1)$ be such that the right-hand side of (4.8) is finite; then by Lemma 4.2, $u \in W^{1,p}(\delta, 1)$ for every $\delta > 0$. If $u \neq \varphi$ in $]0, \delta[$ for a suitable $\delta > 0$, we have

$$\int_0^\delta |u'|^p\,dx = \int_0^\delta a_\varphi(x, u)\,|u'|^p\,dx < +\infty,$$

and so $u \in W^{1,p}(0,1)$. In this case it is enough to take $u_\varepsilon = u$ to satisfy our requirements. Otherwise, let $y_\varepsilon \to 0$ be such that $u(y_\varepsilon) = \varphi(y_\varepsilon)$, and let $x_\varepsilon \to 0$ be such that

$$\lim_{\varepsilon\to 0^+} \frac{|\varphi(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} = \liminf_{x\to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}}. \tag{4.9}$$

Possibly refining the sequence (x_ε) we may assume that $x_\varepsilon < y_\varepsilon$ for every $\varepsilon > 0$. Define now

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if} \quad x > y_\varepsilon \\ \varphi(x) & \text{if} \quad x_\varepsilon \leqslant x \leqslant y_\varepsilon \\ \dfrac{\varphi(x_\varepsilon)}{x_\varepsilon} x & \text{if} \quad x < x_\varepsilon. \end{cases}$$

We have $u_\varepsilon \in W^{1,p}(0, 1)$, $u_\varepsilon \to u$ strongly in $W^{1,1}(0, 1)$, and

$$F_p(u_\varepsilon) \leqslant \int_0^{x_\varepsilon} |u_\varepsilon'|^p \, dx + \int_{y_\varepsilon}^1 a_\varphi(x, u) \, |u'|^p \, dx \leqslant \frac{|\varphi(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} + G(u).$$

Passing to the limit as $\varepsilon \to 0^+$, and recalling (4.9), we obtain (4.8). ∎

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.3, Remark 4.4, and Lemma 4.5 we obtain

$$\bar{F}(u) \leqslant G(u) + L(u) \qquad \text{for every} \quad u \in \mathscr{A}(0, 1).$$

In order to prove the opposite inequality, it is enough to show that the functional $G + L$ is sequentially weakly l.s.c. on $\mathscr{A}(0, 1)$. This is well known in the case

$$\liminf_{x \to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}} = 0;$$

therefore, it remains only to consider the case

$$\liminf_{x \to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}} > 0. \tag{4.10}$$

Let $u, u_\varepsilon \in \mathscr{A}(0, 1)$ be such that $u_\varepsilon \to u$ weakly in $W^{1,1}(0, 1)$; we have to prove that

$$G(u) + L(u) \leqslant \liminf_{\varepsilon \to 0^+} [G(u_\varepsilon) + L(u_\varepsilon)]. \tag{4.11}$$

We may assume the lim inf in the right-hand side of (4.11) is a finite limit; moreover, by (4.2), inequality (4.11) is immediate if $u \in W^{1,p}(0, 1)$. Therefore, we may also assume that $u \notin W^{1,p}(0, 1)$. By Lemma 4.3 and Lemma 4.5, for every $\varepsilon > 0$ there exists $v_\varepsilon \in W^{1,p}(0, 1)$ such that

$$v_\varepsilon(0) = 0 \qquad \|u_\varepsilon - v_\varepsilon\|_{W^{1,1}(0,1)} \leqslant \varepsilon, \qquad F_p(v_\varepsilon) \leqslant G(u_\varepsilon) + L(u_\varepsilon) + \varepsilon. \tag{4.12}$$

Since $v_\varepsilon \in W^{1,p}(0, 1)$ and (4.10) holds, in a suitable interval $]0, x_\varepsilon[$ we have $v_\varepsilon \neq \varphi$; moreover, since $v_\varepsilon \to u$ weakly in $W^{1,1}(0, 1)$ and $u \notin W^{1,p}(0, 1)$, we may assume that $x_\varepsilon \to 0$ and $v_\varepsilon(x_\varepsilon) = \varphi(x_\varepsilon)$. Then, for every $\varepsilon > 0$

$$\frac{|\varphi(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} = \frac{|v_\varepsilon(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} \leqslant \int_0^{x_\varepsilon} |v'_\varepsilon|^p \, dx = \int_0^{x_\varepsilon} a_\varphi(x, v_\varepsilon) |v'_\varepsilon|^p \, dx,$$

which implies, for every $\delta > 0$

$$\begin{aligned} \liminf_{\varepsilon \to 0^+} F_p(v_\varepsilon) &\geqslant \liminf_{\varepsilon \to 0^+} \left[\int_0^{x_\varepsilon} a_\varphi(x, v_\varepsilon) \, |v'_\varepsilon|^p \, dx + \int_\delta^1 a_\varphi(x, v_\varepsilon) \, |v'_\varepsilon|^p \, dx \right] \\ &\geqslant \liminf_{\varepsilon \to 0^+} \frac{|\varphi(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} + \int_\delta^1 a_\varphi(x, u) \, |u'|^p \, dx \\ &\geqslant \liminf_{x \to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}} + \int_\delta^1 a_\varphi(x, u) \, |u'|^p \, dx. \end{aligned}$$

Since $\delta > 0$ is arbitrary, we obtain

$$\liminf_{\varepsilon \to 0^+} F_p(v_\varepsilon) \geqslant G(u) + \liminf_{x \to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}} \geqslant G(u) + L(u)$$

and this, together with (4.12) proves (4.11). ▮

There is a sharpened form of (4.1) which holds.

PROPOSITION 4.6. *For every $u \in \mathscr{A}(0, 1) \backslash W^{1,p}(0, 1)$ with $G(u) < +\infty$ we have*

$$\liminf_{x \to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}} \leqslant \liminf_{x \to 0^+} \frac{|u(x)|^p}{x^{p-1}} \leqslant \limsup_{x \to 0^+} \frac{|u(x)|^p}{x^{p-1}} \leqslant \limsup_{x \to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}}. \tag{4.13}$$

Proof. Let us prove the first inequality in (4.13) by contradiction. Assume

$$\liminf_{x \to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}} > \liminf_{x \to 0^+} \frac{|u(x)|^p}{x^{p-1}} \tag{4.14}$$

and let $x_\varepsilon \to 0$ be such that

$$\lim_{\varepsilon \to 0^+} \frac{|u(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} = \liminf_{x \to 0^+} \frac{|u(x)|^p}{x^{p-1}}. \tag{4.15}$$

From (4.14) and (4.15) it follows that $|\varphi(x_\varepsilon)| > |u(x_\varepsilon)|$ for ε small enough. Set

$$y_\varepsilon = \min\{x \in [x_\varepsilon, 1]: u(x) = \varphi(x)\} \qquad (y_\varepsilon = 1 \text{ if } u \neq \varphi \text{ in} [x_\varepsilon, 1])$$

and assume there exists $\delta > 0$ such that $y_\varepsilon \geqslant \delta$. Then $a_\varphi(x, u) = 1$ in $]x_\varepsilon, \delta[$, so that

$$\int_{x_\varepsilon}^{\delta} |u'|^p \, dx = \int_{x_\varepsilon}^{\delta} a_\varphi(x, u) \, |u'|^p \, dx \leqslant G(u).$$

As $\varepsilon \to 0$ we would obtain

$$\int_0^{\delta} |u'|^p \, dx < +\infty,$$

so that, by Lemma 4.2, $u \in W^{1,p}(0, 1)$ which contradicts the assumption $u \in \mathscr{A}(0, 1) \setminus W^{1,p}(0, 1)$. Therefore, the sequence y_ε (or a subsequence of it) tends to 0, and

$$(y_\varepsilon - x_\varepsilon) \left| \frac{u(y_\varepsilon) - u(x_\varepsilon)}{y_\varepsilon - x_\varepsilon} \right|^p \leqslant \int_{x_\varepsilon}^{y_\varepsilon} |u'|^p \, dx = \int_{x_\varepsilon}^{y_\varepsilon} a_\varphi(x, u) \, |u'|^p \, dx. \tag{4.16}$$

Since $a_\varphi(x, u) \, |u'|^p$ is integrable on $]0, 1[$, setting

$$\omega_\varepsilon = \int_{x_\varepsilon}^{y_\varepsilon} a_\varphi(x, u) \, |u'|^p \, dx,$$

we have $\omega_\varepsilon \to 0$ and , by (4.16),

$$u(y_\varepsilon) \leqslant u(x_\varepsilon) + \omega_\varepsilon^{1/p} |y_\varepsilon - x_\varepsilon|^{(p-1)/p}.$$

Then

$$\frac{|\varphi(y_\varepsilon)|}{y_\varepsilon^{(p-1)/p}} = \frac{|u(y_\varepsilon)|}{y_\varepsilon^{(p-1)/p}} \leqslant \frac{|u(x_\varepsilon)|}{y_\varepsilon^{(p-1)/p}} + \omega_\varepsilon^{1/p} \leqslant \frac{|u(x_\varepsilon)|}{x_\varepsilon^{(p-1)/p}} + \omega_\varepsilon^{1/p}$$

which contradicts (4.14) and (4.15).

Let us prove the last inequality in (4.13) by contradiction. Assume

$$\limsup_{x \to 0^+} \frac{|u(x)|^p}{x^{p-1}} > \limsup_{x \to 0^+} \frac{|\varphi(x)|^p}{x^{p-1}} \tag{4.17}$$

and let $x_\varepsilon \to 0$ be such that

$$\lim_{\varepsilon \to 0^+} \frac{|u(x_\varepsilon)|^p}{x_\varepsilon^{p-1}} = \limsup_{x \to 0^+} \frac{|u(x)|^p}{x^{p-1}}. \tag{4.18}$$

From (4.17) and (4.18) it follows that $|\varphi(x_\varepsilon)| < |u(x_\varepsilon)|$ for ε small enough. As before, if $\varphi(0) \neq 0$, since $u(0) = 0$ and $G(u) < +\infty$, we would obtain

$u \in W^{1,p}(0, 1)$ which contradicts our assumptions. Then $\varphi(0)=0$, so that, setting

$$y_\varepsilon = \max\{x \in [0, x_\varepsilon]: u(x) = \varphi(x)\}$$

$a_\varphi(x, u) = 1$ in $]y_\varepsilon, x_\varepsilon[$. Then, as in the previous part, setting

$$\omega_\varepsilon = \int_{y_\varepsilon}^{x_\varepsilon} a_\varphi(x, u)\, |u'|^p\, dx,$$

we have $\omega_\varepsilon \to 0$ and

$$(x_\varepsilon - y_\varepsilon) \left| \frac{u(x_\varepsilon) - u(y_\varepsilon)}{x_\varepsilon - y_\varepsilon} \right|^p \leqslant \int_{y_\varepsilon}^{x_\varepsilon} |u'|^p\, dx = \int_{y_\varepsilon}^{x_\varepsilon} a_\varphi(x, u)\, |u'|^p\, dx = \omega_\varepsilon,$$

that is,

$$u(x_\varepsilon) \leqslant u(y_\varepsilon) + \omega_\varepsilon^{1/p}\, |x_\varepsilon - y_\varepsilon|^{(p-1)/p}.$$

This implies

$$\frac{|u(x_\varepsilon)|}{x_\varepsilon^{(p-1)/p}} \leqslant \frac{|u(y_\varepsilon)|}{x_\varepsilon^{(p-1)/p}} + \omega_\varepsilon^{1/p} = \frac{|\varphi(y_\varepsilon)|}{x_\varepsilon^{(p-1)/p}} + \omega_\varepsilon^{1/p} \leqslant \frac{|\varphi(y_\varepsilon)|}{y_\varepsilon^{(p-1)/p}} + \omega_\varepsilon^{1/p}$$

which contradicts (4.17) and (4.18). ∎

Remark 4.7. By Proposition 4.6 we may write

$$\bar{F}(u) = \begin{cases} G(u) & \text{if} \quad u \in W^{1,p}(0, 1) \\ G(u) + \liminf\limits_{x \to 0^+} \dfrac{|\varphi(x)|^p}{x^{p-1}} & \text{otherwise.} \end{cases}$$

Moreover, when $|\varphi(x)|^p / x^{p-1}$ tends (as $x \to 0^+$) to a limit (finite or not), taking into account (4.2) and Proposition 4.6 we get

$$\bar{F}(u) = G(u) + \liminf_{x \to 0^+} \frac{|u(x)|}{x^{p-1}} \qquad \text{for every} \quad u \in \mathscr{A}(0, 1).$$

5. Further Remarks

We may consider the Lavrentiev phenomenon in a very abstract framework: given a topological space X, a dense subset $Y \subset X$, and a functional $F: X \to [0, +\infty]$ define

$$\bar{F}_X = \sup\{G: X \to [0, +\infty]: G \text{ is l.s.c., } G \leqslant F \text{ on } X\}$$

$$\bar{F}_Y = \sup\{G: X \to [0, +\infty]: G \text{ is l.s.c, } G \leqslant F \text{ on } Y\}.$$

It is clear that $\bar{F}_X \leqslant \bar{F}_Y$, hence the Lavrentiev term $L(u)$ defined for every $u \in X$ by

$$L(u) = \bar{F}_Y(u) - \bar{F}_X(u) \qquad (L(u) = 0 \quad \text{if} \quad \bar{F}_X(u) = +\infty)$$

turns out to be nonnegative. In particular, $L = \bar{F}_Y - F$ whenever F is l.s.c.

Consider now the case when $X = W^{1,1}(\Omega; \mathbf{R}^m)$, $Y = W^{1,\infty}(\Omega; \mathbf{R}^m)$, and

$$F(u) = \int_\Omega f(x, u, Du)\, dx \qquad (u \in X).$$

Here Ω is a bounded open subset of $\mathbf{R}^n$ with a Lipschitz boundary, X is endowed with the weak convergence , and $f(x, s, z)$ is a nonnegative Borel integrand.

In some situations, it may occur that $L(u) = 0$ whenever $\bar{F}_X(u) < +\infty$, so that the relaxed functional $\bar{F}_Y$ coincides with $\bar{F}_X$. This is the case, for instance, when the integrand f is of Carathéodory type (in the sense of (2.1)) and satisfies a condition of the form

$$c_1(|z|^p + a_1(x)) \leqslant f(x, s, z) \leqslant c_2(|z|^p + |s|^p + a_2(x)) \tag{5.1}$$

with $p > 1$, $0 < c_1 \leqslant c_2$, $a_1, a_2 \in L^1(\Omega)$. Indeed, in this case the following proposition holds.

PROPOSITION 5.1. *The functionals $\bar{F}_Y$ and $\bar{F}_X$ coincide.*

Proof. Since $\bar{F}_X \leqslant \bar{F}_Y$ and since F is finite only on $W^{1,p}(\Omega; \mathbf{R}^m)$, in order to conclude the proof it is enough to show that

$$\bar{F}_Y(u) \leqslant F(u) \qquad \text{for every} \quad u \in W^{1,p}(\Omega; \mathbf{R}^m). \tag{5.2}$$

Let $u \in W^{1,p}(\Omega; \mathbf{R}^m)$ and let (u_h) be a sequence in $Lip[0, 1]$ converging to u strongly in $W^{1,p}(\Omega; \mathbf{R}^m)$. Using the lower semicontinuity of $\bar{F}_Y$ and the fact that by the second inequality of (5.1), F is continuous in the $W^{1,p}$ norm (cf., e.g., [ET]), we get

$$\bar{F}_Y(u) \leqslant \liminf_{h \to +\infty} \bar{F}_Y(u_h) \leqslant \liminf_{h \to +\infty} F(u_h) = F(u)$$

that is (5.2). ∎

Another class of functionals for which the Lavrentiev term $L(u)$ vanishes whenever $\bar{F}_X(u) < +\infty$ is given by all integrals of the form (here $n = m = 1$)

$$F(u) = \int_0^1 f(x, u')\, dx \qquad (u \in W^{1,1}(0, 1)), \tag{5.3}$$

where $f\colon \Omega \times \mathbf{R} \to [0, +\infty]$ is a Borel function such that

$$f(x, \cdot) \text{ is convex and l.s.c. on } \mathbf{R} \text{ for a.e. } x \in \Omega; \tag{5.4}$$

$$\text{there exists } u_0 \in Lip[0, 1] \text{ with } F(u_0) < +\infty. \tag{5.5}$$

Then F is sequentially weakly l.s.c. and the following proposition holds (see De Arcangelis [De]).

PROPOSITION 5.2. *Let $f\colon \Omega \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying* (5.4) *and* (5.5), *and let F be given by* (5.3). *Then we have*

$$\bar{F}_Y = F(u) \qquad \textit{for every} \quad u \in W^{1,1}(0, 1).$$

Proof. By considering the function

$$g(x, z) = f(x, z + u_0'(x))$$

we may reduce ourselves to the case $u_0 = 0$ in (5.5). Moreover, the assumptions made on f imply that the functional F is sequentially weakly l.s.c. on $W^{1,1}(0, 1)$. Therefore we have

$$\bar{F}_Y(u) \geq F(u) \qquad \text{for every} \quad u \in W^{1,1}(0, 1).$$

In order to prove the opposite inequality, fix $u \in W^{1,1}(0, 1)$, and for every $h \in \mathbf{N}$ and $x \in \Omega$ define

$$u_h(x) = u(0) + \int_0^x (u'(t) \wedge h) \vee (-h)\, dt.$$

We have that $u_h \in Lip[0, 1]$ and

$$\begin{aligned}\int_0^1 |u_h' - u'|\, dx &= \int_{\{|u'| > h\}} |h - |u'||\, dx \\ &\leq \int_{\{|u'| > h\}} (h + |u'|)\, dx \leq 2 \int_{\{|u'| > h\}} |u'|\, dx.\end{aligned}$$

Hence $u_h \to u$ strongly in $W^{1,1}(0, 1)$, and so, by the convexity of $f(x, \cdot)$,

$$\begin{aligned}\bar{F}_Y(u) &\leq \liminf_{h \to +\infty} F(u_h) \\ &= \liminf_{h \to +\infty} \left[\int_{\{|u'| \leq h\}} f(x, u')\, dx + \int_{\{u' > h\}} f(x, h)\, dx \right. \\ &\quad \left. + \int_{\{u' < -h\}} f(x, -h)\, dx \right]\end{aligned}$$

$$\leqslant \liminf_{h \to +\infty} \left\{ \int_{\{|u'| \leqslant h\}} f(x, u')\, dx \right.$$

$$\left. + \int_{\{|u'| > h\}} \left[\frac{h}{|u'|} f(x, u') + \left(1 - \frac{h}{|u'|}\right) f(x, 0) \right] dx \right\}$$

$$\leqslant \int_\Omega f(x, u')\, dx + \liminf_{h \to +\infty} \int_{\{|u'| > h\}} f(x, 0)\, dx = \int_\Omega f(x, u')\, dx,$$

where the last equality follows from the fact that $f(x, 0)$ has been supposed integrable and $\text{meas}(\{|u'| > h\}) \to 0$ as $h \to +\infty$. Therefore the proof is completely achieved. ∎

It is known (see Proposition 5.2 and also Clarke and Vinter [CV], Ambrosio, Ascenzi, and Buttazzo [AAB]) that if $n = m = 1$ then in order to have the Lavrentiev phenomenon (that is, $L(u) \neq 0$ for some $u \in X$) the integrand f must depend on all its variables x, s, z. If $n > 1$ and $m = 1$, on the contrary, we may have the Lavrentiev phenomenon even for integrands of the form $f(x, z)$ (see De Arcangelis [De]), whereas if $n > 1$ and $m > 1$ and example in which the Lavrentiev phenomenon occurs has been provited by Bethuel, Brezis, and Coron [BBC] and by Giaquinta, Modica, and Soucek [GMS] with

$$f(s, z) = \begin{cases} |z|^2 & \text{if } \quad |s| = 1 \\ +\infty & \text{otherwise.} \end{cases}$$

In the case $n > 1$, $m > 1$ the Lavrentiev phenomenon may occur even with integrands of the form $f = f(z)$; indeed Müller [Mü] (see also Marcellini [Mar1, Mar2]) showed that if $n = m = 2$, $p \in \,]4/3, 2[$, and

$$F(u) = \int_\Omega |\det Du|\, dx \qquad (u \in W^{1,p}(\Omega, \mathbf{R}^2))$$

with the weak $W^{1,p}$ convergence, one has

$$\bar{F}_X(u) < \bar{F}_Y(u) \qquad \text{for some } u \in W^{1,p}(\Omega, \mathbf{R}^2).$$

The problem of determining whether for $n > 1$, $m > 1$, $f = f(z)$ the Lavrentiev phenomenon can occur in general with l.s.c. functionals of the form

$$F(u) = \int_\Omega f(Du)\, dx$$

is, as far as we know, still open (except in the case $f(z)$ convex, where $L = 0$ under some mild assumptions on f or on Ω).

ACKNOWLEDGMENTS

We acknowledge helpful conversations with Art Heinricher, Bob Jensen, and Stefan Müller. The first author thanks the Department of Mathematics of the EPFL of Lausanne, where this paper was partially written. The research of Mizel was partially supported by the U.S. National Science Foundation under grants DMS 8704530 and DMS 9002562.

REFERENCES

[AAB] L. AMBROSIO, O. ASCENZI, AND G. BUTTAZZO, Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands, *J. Math. Anal. Appl.* **142** (1989), 301–316.

[An] T. S. ANGELL, A note on the approximation of optimal solutions of the calculus of variations, *Rend. Circ. Mat. Palermo* **2** (1979), 258–272.

[BK] J. M. BALL AND G. KNOWLES, A numerical method for detecting singular minimizers, *Numer. Math.* **51** (1987), 181–197.

[BM1] J. M. BALL AND V. J. MIZEL, Singular minimizers for regular one-dimensional problems in the calculus of variations, *Bull. Amer. Math. Soc.* **11** (1984), 143–146.

[BM2] J. M. BALL AND V. J. MIZEL, One-dimensional variational problems whose minimizers do not satisfy the Euler–Lagrange equation, *Arch. Rational Mech. Anal.* **90** (1985), 325–388.

[BBC] F. BETHUEL, H. BREZIS, AND J. M. CORON, Relaxed energies for harmonic maps, preprint.

[Bu] G. BUTTAZZO, Semicontinuity, relaxation and integral representation in the calculus of variations, *in* "Pitman Res. Notes Math. Ser.," Vol. 207, Longman, Harlow, 1989.

[Ce] L. CESARI, "Optimization-Theory and Applications," Springer-Verlag, Berlin, 1983.

[Ca] L. CESARI AND T. S. ANGELL, On the Lavrentiev phenomenon, *Calcolo* **22** (1985), 17–29.

[CV] F. H. CLARKE AND R. B. VINTER, Regularity properties of solutions to the basic problem in the calculus of variations, *Trans. Amer. Math. Soc.* **291** (1985), 73–98.

[Da] A. M. DAVIE, Singular minimizers in the calculus of variations in one dimension, *Arch. Rational Mech. Anal.* **101** (1988), 161–177.

[De] R. DE ARCANGELIS, Some remarks on the identity between a variational integral and its relaxed functional, *Ann. Univ. Ferrara,* **35** (1989), 135–145.

[ET] I. EKELAND AND R. TEMAM, "Convex Analysis and Variational Problems," North-Holland, Amsterdam, 1976.

[GMS] M. GIAQUINTA, G. MODICA, AND J. SOUCEK, The Dirichlet energy of mappings with values into the sphere, preprint, Università di Firenze, Firenze, 1989.

[He] A. C. HEINRICHER, "A Singular Stochastic Control Problem Arising from a Deterministic Problem with Non-Lipschitzian Minimizers," Dissertation, Carnegie Mellon University, 1986.

[HM1] A. C. HEINRICHER AND V. J. MIZEL, The Lavrentiev phenomenon for invariant variational problems, *Arch. Rational Mech Anal.* **102** (1988), 57–93.

[HM2] A. C. HEINRICHER AND V. J. MIZEL, A stochastic control problem with different value functions for singular and absolutely continuous control, *in* "Proceedings, 25th IEEE Conference on Decision and Control, Athens, 1986," pp. 134–139.

[HM3] A. C. HEINRICHER AND V. J. MIZEL, A new example of the Lavrentiev phenomenon, *SIAM J. Control Optim.* **26** (1988), 1490–1503.

[Kn] G. KNOWLES, Finite element approximation to singular minimizers, and applications to cavitation in nonlinear elasticity, *in* "Proceedings, Differential Equations and Mathematical Physics, Birmingham, 1986," pp. 236–247, Lecture Notes in Math., Vol. 1285, Springer-Verlag, New York/Berlin, 1987.

[Io] A. D. IOFFE, On lower semicontinuity of integral functionals, I, II, *SIAM J. Control Optim.* **15** (1977), 521–538 and 991–1000.

[La] M. LAVRENTIEV, Sur quelques problèmes du calcul des variations, *Ann. Math. Pura Appl.* **4** (1926), 107–124.

[Li] F. C. LIU, A Luzin type property of Sobolev functions, *Indiana Univ. Math. J.* **26** (1977), 645–651.

[Lo] P. D. LOEWEN, On the Lavrentiev phenomenon, *Canad. Math. Bull.* **30** (1987), 102–108.

[Ma] B. MANIÀ, Sopra un esempio di Lavrentieff, *Boll. Un. Mat. Ital.* **13** (1934), 146–153.

[Mar1] P. MARCELLINI, On the definition and the lower semicontinuity of certain quasi-convex integrals, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3** (1986), 391–409.

[Mar2] P. MARCELLINI, The stored energy for some discontinuous deformations in nonlinear elasticity, *in* "Essays in Honor of Ennio De Giorgi," pp. 767–786, Birkhäuser, Boston, 1989.

[MM1] M. MARCUS AND V. J. MIZEL, Absolute continuity on tracks and mappings of Sobolev spaces, *Arch. Rational Mech. Anal.* **45** (1972), 294–320.

[MM2] M. MARCUS AND V. J. MIZEL, Every superposition operator mapping one Sobolev space into another is continuous, *J. Funct. Anal.* **33** (1979), 217–229.

[Mi1] V. J. MIZEL, The Lavrentiev phenomenon in both deterministic and stochastic optimization problems, *in* "Proceedings, Integral Functionals in Calculus of Variations, Trieste, 1985," *Rend. Circ. Mat. Palermo Suppl.* **15** (1987), 111–130.

[Mi2] V. J. MIZEL, paper in preparation.

[Mü] S. MÜLLER, personal communication.

[N] P. V. NEGRON MARRERO, A numerical method for detecting singular minimizers of multidimensional problems in nonlinear elasticity, *Numer. Math.* **58** (1990), 135–144.

[Zo] T. ZOLEZZI, Well-posedness and the Lavrentiev phenomenon, preprint, Università di Genova, Genova, 1990.